APPROXIMATIONS OF SUBHOMOGENEOUS ALGEBRAS

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Abstract. Let $n \geq 1$. Recall that a C^*-algebra is said to be n-subhomogeneous if all its irreducible representations have dimension at most n. In this short note, we give various approximation properties characterising n-subhomogeneous C^*-algebras.

1. Introduction

Let A and B be C^*-algebras and let $\phi\colon A \to B$ be a bounded linear map. For each integer $n \geq 1$, we can define maps

$$\phi \otimes \mathrm{id}_{\mathbb{M}_n}\colon A \otimes \mathbb{M}_n \to B \otimes \mathbb{M}_n, \tag{1}$$

where $\mathbb{M}_n$ denote the C^*-algebra of $n \times n$-matrices. We say that ϕ is *n-positive* if $\phi \otimes \mathrm{id}_{\mathbb{M}_n}$ is positive and *n-contractive* if $\phi \otimes \mathrm{id}_{\mathbb{M}_n}$ is contractive. We say that a map is *completely positive (completely contractive)* if it is n-positive (n-contractive) for all $n \geq 1$. As usual, we abbreviate *completely positive* by c.p., *contractive and completely positive* by c.c.p., *unital and completely positive* by u.c.p. and *completely contractive* by c.c. Note that u.c.p. maps are c.c.p. and c.c.p. maps are c.c. by the Stinespring dilation theorem [Sti55, Theorem 1].

Finite-dimensional approximation properties of maps and C^*-algebras play an important role in the study of C^*-algebras. See [BO08] for a comprehensive treatment.

Definition 1.1. A c.c.p. map $\theta\colon A \to B$ is said to be *nuclear* if there exist *finite-dimensional C^*-algebras* F_α and nets of c.c.p. maps $\phi_\alpha\colon A \to F_\alpha$ and $\psi_\alpha\colon F_\alpha \to B$ such that for all $x \in A$,

$$||(\theta - \psi_\alpha \circ \phi_\alpha)(x)|| \to 0 \quad \text{as } \alpha \to \infty. \tag{2}$$

Definition 1.2. A C^*-algebra A is said to be *nuclear* if the identity map $\mathrm{id}_A\colon A \to A$ is nuclear and *exact* if there exists a faithful representation $\pi\colon A \to B(H)$ which is nuclear.

The following is the standard example.

Example 1.3. Let Γ be a countable discrete group. Then the *reduced group C^*-algebra* $C^*_\lambda(\Gamma)$ is nuclear if and only if Γ is amenable. In particular, the reduced group C^*-algebra $C^*_\lambda(F_2)$ of a free group on two generators is non-nuclear. See [BO08, Section 2.6.].

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It is well known that a C^*-algebra is nuclear if and only if the identity map is a point-norm limit of finite-rank c.c.p. maps. On the other hand, it was shown by De Cannière and Haagerup that the identity map on $C^*_\lambda(F_2)$ is a point-norm limit of finite-rank c.c. maps (cf. [DCH85, Corollary 3.11]). This is in contrast to the following theorem of Smith, which says that we recover nuclearity if we insist that the finite-rank c.c. maps to factor through finite-dimensional C^*-algebras.

Theorem 1.4 (Smith [Smi85])**.** *A C^*-algebra A is nuclear if and only if there exist* finite-dimensional C^*-algebras F_α *and nets of c.c. maps $\phi_\alpha\colon A \to F_\alpha$ and $\psi_\alpha\colon F_\alpha \to A$ such that for all $x \in A$,*

$$||(\mathrm{id}_A - \psi_\alpha \circ \phi_\alpha)(x)|| \to 0 \quad \textit{as } \alpha \to \infty. \tag{3}$$

□

All *abelian* C^*-algebras are nuclear. In fact, the standard proof based on partition of unities shows that one can take the finite-dimensional C^*-algebras F_α's to be abelian and c.c.p. maps ϕ_α's to be $*$-homomorphisms (cf. [BO08, Proposition 2.4.2]).

Our investigation grew out of the following simple question.

Question 1.5. Suppose that the there exist finite-dimensional *abelian* C^*-algebras F_α and nets of c.c.p. maps $\phi_\alpha\colon A \to F_\alpha$ and $\psi_\alpha\colon F_\alpha \to A$ such that for all $x \in A$,

$$||(\mathrm{id}_A - \psi_\alpha \circ \phi_\alpha)(x)|| \to 0 \quad \text{as } \alpha \to \infty. \tag{4}$$

Can we conclude that A is abelian? Can we still conlcude that A is abelian if we assume that the maps ϕ_α and ψ_α are only c.c.?

Not surprisingly, the answer is YES. In this paper we prove the following.

Let $n \geq 1$. Recall that a C^*-algebra is said to be *n-subhomogeneous* if all its irreducible representations have dimension $\leq n$. Clearly, a C^*-algebra is abelian if and only if it is 1-subhomogeneous. A finite-dimensional C^*-algebra is n-subhomogeneous if and only if it is a finite product of matrix algebras $\mathbb{M}_k$ of size $k \leq n$.

Theorem 1.6. *Let A be a C^*-algebra and let $n \geq 1$ be an integer. Then the following are equivalent.*

(1) The C^*-algebra A is n-subhomogeneous.

(2) There exist nets of $*$-homomorphisms *$\phi_\alpha\colon A \to F_\alpha$ and c.c.p. maps $\psi_\alpha\colon F_\alpha \to A$, with F_α finite dimensional and n-subhomogeneous, such that for all $x \in A$,*

$$||x - \psi_\alpha \circ \phi_\alpha(x)|| \to 0 \quad \textit{as } \alpha \to \infty. \tag{5}$$

(3) There exist nets of c.c.p. maps $\phi_\alpha\colon A \to F_\alpha$ and $\psi_\alpha\colon F_\alpha \to A$, with F_α (finite dimensional and) n-subhomogeneous, such that for all $x \in A$,

$$||x - \psi_\alpha \circ \phi_\alpha(x)|| \to 0 \quad \textit{as } \alpha \to \infty. \tag{6}$$

(4) *There exist nets of c.c. maps* $\phi_\alpha \colon A \to F_\alpha$ *and* $\psi_\alpha \colon F_\alpha \to A$*, with* F_α *(finite dimensional and)* n*-subhomogeneous, such that for all* $x \in A$*,*

$$||x - \psi_\alpha \circ \phi_\alpha(x)|| \to 0 \quad \textit{as } \alpha \to \infty. \tag{7}$$

Proof. The nontrivial implications are (1) ⇒ (2), (3) ⇒ (1) and (4) ⇒ (1). See Theorem 1.7 below. □

Our proof is based on the solution of the Choi conjecture [Cho72], due to Tomiyama [Tom82] and Smith [Smi83], and a contractive analogue of the Choi conjecture (see Theorem 2.15). See also [Loe75] and [HT83].

The following is a summary of the results.

Theorem 1.7. *Let* A *be a* C^**-algebra and let* $n \geq 1$ *be an integer. Then the following are equivalent.*

(1) *The* C^**-algebra* A *is* n*-subhomogeneous.*
(2) *There exist nets of* $*$-homomorphisms $\phi_\alpha \colon A \to F_\alpha$ *and c.c.p. maps* $\psi_\alpha \colon F_\alpha \to A$*, with* F_α *finite dimensional and* n*-subhomogeneous, such that for all* $x \in A$*,*

$$||x - \psi_\alpha \circ \phi_\alpha(x)|| \to 0 \quad \textit{as } \alpha \to \infty. \tag{8}$$

(3) *There exist nets of* n*-positive maps* $\phi_\alpha \colon A \to F_\alpha$ *and* $\psi_\alpha \colon F_\alpha \to A$*, with* F_α *finite dimensional and* n*-subhomogeneous, such that for all* $x \in A$*,*

$$||x - \psi_\alpha \circ \phi_\alpha(x)|| \to 0 \quad \textit{as } \alpha \to \infty. \tag{9}$$

(4) *All* n*-positive maps with domain and/or range* A *are completely positive.*
(5) *There exist nets of* n*-contractive maps* $\phi_\alpha \colon A \to F_\alpha$ *and* $(n+1)$*-contractive maps* $\psi_\alpha \colon F_\alpha \to A$*, with* F_α *finite dimensional and* n*-subhomogeneous, such that for all* $x \in A$*,*

$$||x - \psi_\alpha \circ \phi_\alpha(x)|| \to 0 \quad \textit{as } \alpha \to \infty. \tag{10}$$

(6) *All* n*-contractive maps with range* A *are completely contractive.*

Proof. See Theorems 2.9, 2.11 and 2.16. □

In section 3, we show that even weaker approximation property characterises abelianness. See Theorem 3.2.

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2. Subhomogeneous Algebras

Let $n \geq 1$ be an integer.

Definition 2.1. We say that a C^*-algebra is *n-subhomogeneous* if all its irreducible representations have dimension $\leq n$.

Example 2.2. A finite-dimensional C^*-algebra is n-subhomogeneous if and only if it is a finite product of matrix algebras $\mathbb{M}_k$ with $k \leq n$.

In the following, we summarise some well-known properties of n-subhomogeneous C^*-algebras. See also [Bla06, IV.1.4].

Proposition 2.3. *Let $n \geq 1$ be an integer. The following statements hold.*

1. *A C^*-subalgebra of a n-subhomogeneous algebra is n-subhomogeneous.*
2. *A C^*-algebra A is n-subhomogeneous if and only if $A \subseteq \mathbb{M}_n(B)$ for some abelian C^*-algebra B.*
3. *A C^*-algebra A is n-subhomogeneous if and only if its bidual A^{**} is n-subhomogeneous as a C^*-algebra.*
4. *The product/sum of C^*-algebras A_i, $i \in I$, is n-subhomogeneous if and only if each A_i, $i \in I$, is n-subhomogeneous.*
5. *Let $0 \to I \to A \to B \to 0$ be an extension of C^*-algebras. Then A is n-subhomogeneous if and only if I and B are n-subhomogeneous.*

Proof.
(1) Follows from [Ped79, Proposition 4.1.8].
(2) If A is n-subhomogeneous, then $A \subseteq \mathbb{M}_n(l^\infty(\widehat{A}))$, where $\widehat{A}$ denote the set of unitary equivalence classes of irreducible representations of A. The other direction follows from (1).
(3) Since $A \subseteq A^{**}$, if A^{**} is n-subhomogeneous, then so is A by (1). Conversely, if A is n-subhomogeneous, then writing $A \subseteq \mathbb{M}_n(B)$ with B abelian using (2), we see that $A^{**} \subseteq \mathbb{M}_n(B^{**})$. We are done by (2), since B^{**} is abelian.
(4) Follows from (2).
(5) Follows from (3) and (4), since $A^{**} \cong I^{**} \oplus B^{**}$.

□

The structure of n-subhomogeneous C^*-algebras can be rather complicated (see for instance [TT61]). However, the situation for von Neumann algebras is well-known to be very simple.

Lemma 2.4. *Suppose that a von Neumann algebra M is n-subhomogeneous as a C^*-algebra. Then*

$$M \cong \prod_{k \leq n} \mathbb{M}_k(B_k), \tag{11}$$

where B_k, $k \leq n$, are abelian *von Neumann algebras.*

Proof. Since exactness passes to C^*-subalgebras, n-subhomogeneous algebras are exact by Proposition 2.3(2). Now [BO08, Proposition 2.4.9] completes the proof. □

Subhomogeneous algebras are type I, hence nuclear (cf. [BO08, Proposition 2.7.7]). Scrutinizing the proof, we see that the following slightly stronger approximation property holds. We consider the unital case first.

Theorem 2.5. *Let $n \geq 1$ and let A be a unital n-subhomogeneous C^*-algebra. Then there exist finite-dimensional n-subhomogeneous C^*-algebras F_α and nets of unital* $*$-homomorphisms *$\phi_\alpha \colon A \to F_\alpha$ and u.c.p. maps $\psi_\alpha \colon F_\alpha \to A$ such that for all $x \in A$,*

$$||x - \psi_\alpha \circ \phi_\alpha(x)|| \to 0 \quad \text{as } \alpha \to \infty. \tag{12}$$

Definition 2.6. Let A and B be unital C^*-algebras. We say a u.c.p. map $\theta \colon A \to B$ is *n-factorable* if it can be expressed as a composition $\theta = \psi \circ \phi$, where $\phi \colon A \to F$ is a unital $*$-homomorphism and $\psi \colon F \to B$ a u.c.p. map and F a finite-dimensional n-subhomogeneous C^*-algebra.

Lemma 2.7. *For any unital C^*-algebras A and B, the set of n-factorable maps $A \to B$ is convex.*

Proof. The proof of [BO08, Lemma 2.3.6] applies. □

Lemma 2.8. *Let F be a finite-dimensional C^*-algebra and let A be a unital C^*-algebra. Then u.c.p. maps $F \to A^{**}$ can be approximated by u.c.p. maps $F \to A$ in the point-ultraweak topology.*

Proof. We claim that c.p. maps $F \to A$ correspond bijectively to positive elements in $F \otimes A$. Indeed, for matrix algebras this is a well-known result of Arveson (cf. [BO08, Proposition 1.5.12]). The general case follows, since F is a finite product of matrix algebras and for c.p. maps finite products and finite coproducts coincide. Since positive elements in $F \otimes A$ is ultraweakly dense in the positive elements in $F \otimes A^{**} \cong (F \otimes A)^{**}$, we see that c.p. maps $F \to A^{**}$ can be approximated by c.p. maps $F \to A$ in the point-ultraweak topology.

Let $\psi \colon F \to A^{**}$ be a u.c.p. map and let $\psi_\lambda \colon F \to A$ be a net of c.p. maps converging to ψ in the point-ultraweak topology. Since $\psi_\lambda(1_F) \in A$ is a net converging to 1_A weakly, by passing to convex linear combinations, we may assume that $\psi_\lambda(1_F)$ converges to 1_A in norm and passing to a subnet we may assume that $\psi_\lambda(1_F)$ is invertible. Then $\tilde{\psi}_\lambda(x) := \psi_\lambda(1_F)^{-1/2}\psi_\lambda(x)\psi_\lambda(1_F)^{-1/2}$, $x \in F$, gives the required approximation. □

Proof of Theorem 2.5. For $n = 1$ and A unital abelian, the claim follows from the classical proof of nuclearity for abelian algebras (cf. [BO08, Proposition 2.4.2]).

For general n, first assume that A is of the form

$$\prod_{k \leq n} \mathbb{M}_k(A_k), \tag{13}$$

where A_k, $k \leq n$, are unital abelian C^*-algebras. Then the claim is easily deduced from the case $n = 1$.

Now we consider a general n-subhomogeneous A. By Proposition 2.3(3) and Lemma 2.4, the bidual A^{**} is of the form (13), hence $\mathrm{id}_{A^{**}}$ can be approximated by n-factorable maps $A^{**} \to A^{**}$ in point-norm topology. Then by Lemma 2.8, id_A can be approximated by n-factorable maps in

point-weak topology. Now Lemma 2.7 and [BO08, Lemma 2.3.4] completes the proof. □

As a corollary we obtain the following.

Theorem 2.9. *Let A be a C^*-algebra and let $n \geq 1$ be an integer. Then A is n-subhomogeneous if and only if there exist nets of* $*$-homomorphisms *$\phi_\alpha \colon A \to F_\alpha$ and c.c.p. maps $\psi_\alpha \colon F_\alpha \to A$, with F_α finite-dimensional n-subhomogeneous, such that for all $x \in A$,*

$$||x - \psi_\alpha \circ \phi_\alpha(x)|| \to 0 \quad \text{as } \alpha \to \infty. \tag{14}$$

Proof. (⇒) The unitization A^+ is n-subhomogeneous, hence id_{A^+} can be approximated as in Theorem 2.5. Now restrict ϕ_α to A and replace ψ_α by $e_\beta \psi_\alpha e_\beta$, where e_β is an approximate unit in A (cf. [BO08, Exercise 2.3.4]).

(⇐) Clearly A is a C^*-subalgebra of $\prod_\alpha F_\alpha$. By Proposition 2.3(4 & 1), A is n-subhomogeneous. □

It turns out that much weaker approximation properties imply n-subhomogeneity. Our first result depends on the following.

Theorem 2.10 (Choi, Tomyama, Smith)**.** *Let A and B be C^*-algebras and let $n \geq 1$ be an integer. Then all n-positive maps $A \to B$ are completely positive if and only if A or B is n-subhomogeneous.*

Proof. Choi proved the sufficiency (⇐) for $A = \mathbb{M}_n(D)$ (cf. [Cho72, Theorem 8]) and $B = \mathbb{M}_n(D)$ (cf. [Cho72, Theorem 7]) with D abelian and conjectured the necessity (⇒). A complete proof is obtained by Tomiyama (cf. [Tom82, Theorem 1.2]). The necessity was also proved by Smith (cf. [Smi83, Theorem 3.1]). □

Theorem 2.11. *Let A be a C^*-algebra and let $n \geq 1$ be an integer. Then the following are equivalent.*

1. *There exist nets of n-positive maps $\phi_\alpha \colon A \to F_\alpha$ and $\psi_\alpha \colon F_\alpha \to A$, with F_α finite-dimensional n-subhomogeneous, such that for all $x \in A$,*
$$||x - \psi_\alpha \circ \phi_\alpha(x)|| \to 0 \quad \text{as } \alpha \to \infty. \tag{15}$$
2. *All n-positive maps with domain A are completely positive.*
3. *All n-positive maps with range A are completely positive.*
4. *All n-positive maps $A \to A$ are completely positive.*
5. *The C^*-algebra A is n-subhomogeneous.*

Proof. Let $\phi_\alpha \colon A \to F_\alpha$ and $\psi_\alpha \colon F_\alpha \to A$ be a n-positive approximation of id_A in point-norm topology, with F_α (finite-dimensional) n-subhomogeneous. Let $\theta \colon A \to B$ be a n-positive map. Then $\theta \circ \psi_\alpha \colon F_\alpha \to B$ is a n-positive map with n-subhomogeneous domain, hence c.p. map by Theorem 2.10 and $\phi_\alpha \colon A \to F_\alpha$ is a n-positive map with n-subhomogeneous range, hence also c.p. Since θ is the point-norm limit of $(\theta \circ \psi_\alpha) \circ \phi_\alpha$, we see that θ is c.p. Hence (1) ⇒ (2). Similarly (1) ⇒ (3).

The implications (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4) are clear and the implication (4) $\Rightarrow$ (5) is immediate from Theorem 2.10. Finally, the implication (5) $\Rightarrow$ (1) follows from Theorem 2.9. $\square$

Remark 2.12. The sufficiency in Theorem 2.10 can be deduced from the cases $A = \mathbb{M}_n$ (cf. [Cho72, Theorem 6]) and $B = \mathbb{M}_n$ (cf. [Cho72, Theorem 5]) using Theorem 2.11.

Now we consider the contractive analogue.

Lemma 2.13. *Let $\tau_n \colon \mathbb{M}_n \to \mathbb{M}_n$, $n \geq 1$, denote the transpose map and let $m \geq 1$. Then*

$$||\tau_n \otimes \mathrm{id}_{\mathbb{M}_m} : \mathbb{M}_n \otimes \mathbb{M}_m \to \mathbb{M}_n \otimes \mathbb{M}_m|| = \min\{m, n\}. \tag{16}$$

Proof. For $n \leq m$, this is well-known. The general case follows from the identity

$$(\tau_n \otimes \tau_m) \circ (\tau_n \otimes \mathrm{id}_{\mathbb{M}_m}) = \mathrm{id}_{\mathbb{M}_n} \otimes \tau_m, \tag{17}$$

since $\tau_n \otimes \tau_m$ can be identified with τ_{nm}, hence an isometry. $\square$

Corollary 2.14. *Let $n \geq 2$ be an integer. Then the map*

$$\frac{1}{n-1} \tau_n \colon \mathbb{M}_n \to \mathbb{M}_n \tag{18}$$

is $(n-1)$-contractive, but not n-contractive.

As a corollary, we obtain the following contractive analogue of Theorem 2.10. Note that we have only one of the directions (cf. [Loe75, Theorem C]).

Theorem 2.15. *Let A and B be C^*-algebras and let $n \geq 1$ be an integer. If A and B both admit irreducible representations of dimension $\geq (n+1)$, then there exists an n-contractive map $A \to B$ which is not $(n+1)$-contractive.*

Proof. The proof of [Smi83, Theorem 3.1] applies. See also [Tom82, Lemma 1.1 & Theorem 1.2] $\square$

Theorem 2.16. *Let A be a C^*-algebra and let $n \geq 1$ be an integer. Then the following are equivalent.*

(1) There exist nets of n-contractive maps $\phi_\alpha \colon A \to F_\alpha$ and $(n+1)$-contractive maps $\psi_\alpha \colon F_\alpha \to A$, with F_α finite-dimensional n-subhomogeneous, such that for all $x \in A$,

$$||x - \psi_\alpha \circ \phi_\alpha(x)|| \to 0 \quad \text{as } \alpha \to \infty. \tag{19}$$

(2) All n-contractive maps with range A are $(n+1)$-contractive.
(3) All n-contractive maps with range A are completely contractive.
(4) All n-contractive maps $A \to A$ are $(n+1)$-contractive.
(5) All n-contractive maps $A \to A$ are completely contractive.
(6) The C^*-algebra A is n-subhomogeneous.

Proof. We prove the implications

$$\begin{tikzcd} (1) \arrow[d, Rightarrow] & (6) \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[dl, Rightarrow] & (3) \arrow[d, Rightarrow] \\ (2) \arrow[r, Rightarrow] & (4) \arrow[u, Rightarrow] & (5) \arrow[l, Rightarrow] \end{tikzcd} . \tag{20}$$

The implications (3) $\Rightarrow$ (5), (5) $\Rightarrow$ (4) and (2) $\Rightarrow$ (4) are clear. The implication (4) $\Rightarrow$ (6) follows from Theorem 2.15 and the implication (6) $\Rightarrow$ (1) follows from Theorem 2.9. The implication (6) $\Rightarrow$ (3) follows from [Smi83, Theorem 2.10]. Since (3) $\Rightarrow$ (2) is clear, we also have (6) $\Rightarrow$ (2).

Finally, the implication (1) $\Rightarrow$ (2) is analogous to the proof of Theorem 2.11((1) $\Rightarrow$ (3)). $\square$

Compare with the Loebl conjecture [Loe75], solved affirmatively by Huruya-Tomiyama [HT83] and Smith [Smi83].

Remark 2.17. Note that the statement

(7) All n-contractive maps with *domain* A are $(n+1)$-contractive.

is *not* equivalent to the conditions in Theorem 2.16 in general (cf. [Loe75, Theorem C]).

3. The Abelian Case

Specialising to $n = 1$ in Theorem 2.11, we obtain the following.

Theorem 3.1. *Let A be a C^*-algebra. Suppose that there exist nets of contractive positive maps $\phi_\alpha \colon A \to F_\alpha$ and $\psi_\alpha \colon F_\alpha \to A$, with F_α* abelian, *such that for all $x \in A$,*

$$||x - \psi_\alpha \circ \phi_\alpha(x)|| \to 0 \quad \textit{as } \alpha \to \infty. \tag{21}$$

Then A is abelian.

We give an alternative proof.

Proof. First note that ϕ_α and ψ_α are c.c.p. (cf. [Sti55, Theorem 3 & 4]).

Unitizing if necessary, we may assume that A is unital. Let A^{opp} denote the opposite algebra of A. Then the canonical map $\iota \colon A \to A^{\mathrm{opp}}$ is a pointwise limit of c.c.p. maps $\psi_\alpha^{\mathrm{opp}} \circ \phi_\alpha \colon A \to F_\alpha \cong F_\alpha^{\mathrm{opp}} \to A^{\mathrm{opp}}$, hence a c.c.p. map. Moreover, since ι sends unitaries to unitaries, its multiplicative domain is the whole of A. It follows that ι is a $*$-homomorphism and A is abelian.[1] $\square$

In fact, the following is true.

[1]Walter's 3×3 trick shows that if the canonical map $\iota \colon A \to A^{\mathrm{opp}}$ is 3-positive then A is abelian (cf. [Wal03]).

Theorem 3.2. *Let $\theta\colon A \to B$ be an* injective *$*$-homomorphism. Suppose that there exist nets of contractive maps $\phi_\alpha\colon A \to F_\alpha$ and 2-contractive maps $\psi_\alpha\colon F_\alpha \to B$, with F_α* abelian*, such that for all $x \in A$,*

$$||(\theta - \psi_\alpha \circ \phi_\alpha)(x)|| \to 0 \quad \text{as } \alpha \to \infty. \tag{22}$$

Then A is abelian.

Our main tool is the following beautiful theorem of Takesaki. Let A_1 and A_2 be C^*-algebras. The *injective cross-norm* of A_1 and A_2 is defined by

$$||x||_\lambda \coloneqq \sup |(\varphi_1 \otimes \varphi_2)(x)| \tag{23}$$

where φ_1 and φ_2 run over all contractive linear functionals of A_1 and A_2, respectively. The *injective C^*-cross-norm* of A_1 and A_2 is defined by

$$||x||_{\min} \coloneqq \sup ||(\pi_1 \otimes \pi_2)(x)|| \tag{24}$$

where π_1 and π_2 run over all unitary representations of A_1 and A_2, respectively.

Note that we always have $|| \cdot ||_\lambda \leq || \cdot ||_{\min}$ (cf. [Tak02, IV.4(12)]).

Theorem 3.3 (Takesaki [Tak02, Theorem IV.4.14])**.** *Let A_1 and A_2 be C^*-algebras. Then the norms $|| \cdot ||_{\min}$ and $|| \cdot ||_\lambda$ on $A_1 \otimes A_2$ are equal if and only if A_1 or A_2 is abelian.* □

Equipped with Takesaki's theorem, we can now mimic the proof that nuclear C^*-algebras are tensor-nuclear[2] (cf. [BO08, Proposition 3.6.12]).

Proof of Theorem 3.2. We show that for any $x \in A \otimes \mathbb{M}_2$, we have $||x||_{\min} \leq ||x||_\lambda$. Then Theorem 3.3 completes the proof.

Let $x \in A \otimes \mathbb{M}_2$. The map

$$\theta \otimes_{\min} \mathrm{id}_{\mathbb{M}_2}\colon A \otimes_{\min} \mathbb{M}_2 \to B \otimes_{\min} \mathbb{M}_2 \tag{25}$$

is an injective $*$-homomorphism, hence an isometry. Thus

$$||x||_{A\otimes_{\min}\mathbb{M}_2} = ||\theta \otimes \mathrm{id}_{\mathbb{M}_2}(x)||_{B\otimes_{\min}\mathbb{M}_2}. \tag{26}$$

Writing x as the sum of elementary tensors, we see that

$$||(\theta - \psi_\alpha \circ \phi_\alpha) \otimes \mathrm{id}_{\mathbb{M}_2}(x)||_{B\otimes_{\min}\mathbb{M}_2} \to 0 \quad \text{as } \alpha \to \infty. \tag{27}$$

Hence

$$||x||_{A\otimes_{\min}\mathbb{M}_2} = \lim_{n\to\infty} ||(\psi_\alpha \circ \phi_\alpha) \otimes \mathrm{id}_{\mathbb{M}_2}(x)||_{B\otimes_{\min}\mathbb{M}_2}. \tag{28}$$

On the other hand, it follows from the assumptions that the maps

$$\phi_\alpha \otimes_\lambda \mathrm{id}_{\mathbb{M}_2}\colon A \otimes_\lambda \mathbb{M}_2 \to F_\alpha \otimes_\lambda \mathbb{M}_2 \quad \text{and} \tag{29}$$

$$\psi_\alpha \otimes_{\min} \mathrm{id}_{\mathbb{M}_2}\colon F_\alpha \otimes_{\min} \mathbb{M}_2 \to B \otimes_{\min} \mathbb{M}_2 \tag{30}$$

are contractions and since F_α is abelian, the canonical map

$$F_\alpha \otimes_{\min} \mathbb{M}_2 \to F_\alpha \otimes_\lambda \mathbb{M}_2 \tag{31}$$

[2] It is actually closer to the proof of the fact that exact C^*-algebras with Lance's weak expectation property are tensor-nuclear.

is an isometry by Theorem 3.3. Hence, we have

$$\begin{aligned} ||(\psi_\alpha \circ \phi_\alpha) \otimes \mathrm{id}_{\mathbb{M}_2}(x)||_{B\otimes_{\min}\mathbb{M}_2} &\le ||\phi_\alpha \otimes \mathrm{id}_{\mathbb{M}_2}(x)||_{F_\alpha\otimes_{\min}\mathbb{M}_2} && (32)\\ &= ||\phi_\alpha \otimes \mathrm{id}_{\mathbb{M}_2}(x)||_{F_\alpha\otimes_{\lambda}\mathbb{M}_2} && (33)\\ &\le ||x||_{A\otimes_\lambda \mathbb{M}_2}. && (34)\end{aligned}$$

It follows that $||x||_{\min} \le ||x||_\lambda$. □

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